

Magnetic monopoles, squashed 3-spheres and gravitational instantons from exotic \mathbb{R}^4

Torsten Asselmeyer-Maluga*

German Aero Space Center, Rutherfordstr. 2, 12489 Berlin

Jerzy Król†

University of Silesia, Institute of Physics, ul. Uniwersytecka 4, 40-007 Katowice

We show that, in some limit, gravitational instantons correspond to exotic smooth geometry on \mathbb{R}^4 . The geometry of this exotic \mathbb{R}^4 represents magnetic monopoles of Polyakov-'t Hooft type and the BPS condition allows for the generation of the charges from the gravitational sources of exotic \mathbb{R}^4 . Higgs field is, as usual, present in the monopole configurations. We indicate some possible scenarios in cosmology, particle physics and condensed matter where pure $SU(2)$ Yang-Mills theory on the exotic \mathbb{R}^4 acquires non-zero mass for its gauge boson when the smoothness is changed to the standard \mathbb{R}^4 and, then, it is described by Yang-Mills Higgs theory. The derivation of the results is based on the quasi-modularity of the expressions and the geometry of foliations in the limit.

I. INTRODUCTION

Foliations are quite remarkable objects in geometry and topology. They refine the structure of fiber bundles and fibrations in some cases. They are purely classical geometric 'tiling' of a manifold by (lower-dimensional) leaves. On the other hand, from the point of view of the geometry of their spaces of leaves, foliations are 'quantum' objects and were recognized by Connes as the basic instances of the non-commutative geometries. He assigned to every foliation a (convolution) C^* -algebra of operators which in many cases is non-commutative. But when (quantum) field theory is formulated on a manifold M then, usually, one forgets the foliations and 'purely classical' geometric structures on M overwhelm our practice and thinking. This is rather natural since physical results do not depend on foliations. Is it true, indeed? In general relativity (GR) on Lorentzian manifolds the global hyperbolic structure is nothing but the choice of certain 'foliation' of 4-manifold via 3-dimensional slices (leaves). Many other results, especially in GR, rely heavily on such foliations of spacetime (for example ADM). There are also well known examples of the foliations of compact spheres and tori, like Reeb and Kronecker foliations, which also appear in some physical contexts. We see that indeed there are many instances of the relevance of such (codimension-one) foliations in physics, but the point is that these are, in some sense, trivial. The invariants distinguishing certain equivalence classes of codimension-one foliations of a compact manifold M are the Godbillon-Vey (GV) invariants. It was, soon after, shown by Thurston that GV invariant naturally take values in $H^3(M, \mathbb{R})$. It distinguishes the cobordism classes of codimension-one foliations of M . All the foliations of compact manifolds mentioned above have vanishing GV invariant hence are trivial from that point of view. Our main concern in this work is the physical meaning of codimension-one foliations of the 3-sphere (or other compact 3-manifolds) with non-vanishing GV class. This is far less obvious that these have any relevance to physics. To grasp more concretely what are codimension-one foliations with $GV \neq 0$ let us turn to the connection with quantum physics. Even though not all non-commutative C^* -algebras are derived from some foliations, almost all physically relevant cases are supposed to be. Namely the factor I algebra is generated by the Reeb foliation whereas the factor II by the Kronecker foliations of tori but both have vanishing GV class. It is the factor III algebra which is generated by foliations with $GV \neq 0$. So, on the one hand, foliations are classical geometric structures whereas on the other hand, foliations are (extensions) of quantum mechanical algebras of observables. This property makes them particularly well-suited for exploring the regime of physical theories which are formulated on manifolds and which aim towards the description of quantum realm. This is the case of almost all quantum field theories.

So, we follow the recent proposition to use the codimension-one foliations of certain 3-manifolds with $GV \neq 0$ for exploring the intermediate region of classical and quantum field theories. On the other hand, classical solutions of general relativity in diverse dimensions have been analyzed from various perspectives for many years, and the instanton-like configurations appeared as special in approaching a quantum theory of gravity. They bear potential role in the determination of transition amplitudes in quantum gravity being at the same time classical solutions of GR and they are also well-suited towards exploring the intermediate regime between classical and quantum gravity. In this paper we show that, indeed, the relation between foliations with $GV \neq 0$ and gravitational instantons, can broaden our understanding of some 'quantum' limit of GR.

*Electronic address: torsten.asselmeyer-maluga@dlr.de

†Electronic address: iriking@wp.pl

Gravitational instantons, in general, might be quite similar to the case of instantons in gauge theory. Instantons are the self-dual finite Euclidean-action configurations which minimize the action. The determination of the complete set of instanton configurations is crucial for the evaluation of the path integral of gauge theory in the semi-classical approximation. The functional integral reduces then to an integral over the instanton moduli space in each sector of the topological charge k [44]. Even though, the path integral over instanton moduli space is hardly done directly, the localization technique, along with the tools from non-commutative geometry, allow for the successful computations and the results obtained in this way, perfectly match Seiberg-Witten theory (SW) [1].

So, extending the path integral over gravitation, one could expect that 'gravitational instantons' play a similar substantial role in quantum gravity. However, quantum gravity in 4 dimensions as the attempt to quantize general relativity does not exist as any quantum field theory built on the base of GR. This is mainly due to the appearance of various divergences which may suggest that Einstein's general relativity in the present form is rather a low-energy or large-distance approximation to some more fundamental theory. On the other hand 10-d general relativity appears as classical approximation derived from the β -function of superstring theory. Moreover, gravitons are necessarily present in the spectra of the perturbative string theory. These important features of higher dimensional theories if applicable somehow in dimension 4 would be a crucial step in determining 4-d QG.

The Euclidean gravitational action is not positive-definite in general, even for real positive-definite metrics, however, one can still evaluate the functional integral by first looking for non-singular stationary points of the action functional and expanding about them. These critical points are finite action solutions to the vacuum Einstein equations by self-dual or anti-self-dual complete, non-singular, and positive-definite metrics [2], so they are gravitational analogues of Yang-Mills instantons, i.e. gravitational instantons [3]. In general it is shown [4] that the self-dual or anti-self-dual metrics are local minima of the action among metrics with zero scalar curvature. Facing all these facts, we try to assign some gravitational instantons to exotic \mathbb{R}^4 in such a way that this would help evaluating dominant contributions to the gravitational path integral. As Witten argued in [5] the physical relevant gravitational instantons are represented precisely by exotic higher dimensional spheres S^n , $n = 7, 11$. This result was derived from the description of the group of large diffeomorphisms of S^{n-1} in terms of exotic smoothness on S^n . Since there are no exotic spheres in dimensions 1, 2, 3, 5, 6, 12 and there exist, in particular, exotic S^7 and S^{11} , Witten showed how these govern gravitational anomalies in the theories relevant to string theory in dimension 6 and 10. These exotic S^7 and S^{11} serve as gravitational instantons.

One would like to recognize similarly the case of S^4 where one expects physical relevance as well. However, it is not known currently whether an exotic S^4 exists (smooth 4-d Poincare conjecture). Even though they exist, their connections with the large diffeomorphism group of S^3 , $\text{Diff}(S^3)$, would not follow the pattern from higher dimensions (see e.g. [6]) employed by Witten. Moreover, exotic S^4 's would be supported rather on Euclidean asymptotically flat 4-geometries on \mathbb{R}^4 . This means that induced smoothness on \mathbb{R}^4 would be localized in the exotic 4-disk. This, in turn, means that known small exotic \mathbb{R}^4 's are not of this type. Currently the smooth 4-d Poincare conjecture is not resolved yet. On the other hand it is confirmed that there exists a plenty of exotic smooth structures on Euclidean \mathbb{R}^4 . Hence, in this paper we analyze the case of exotic smooth \mathbb{R}^4 's grouped in a radial family (first described in [7]) and their relation with gravitational instantons. Even though, any direct derivation of this kind of results is not possible at present, the interesting pattern emerges via indirect reasoning based on quasi-modularity.

Recently it was shown that (cobordism classes of) codimension-one foliations of S^3 with non-zero GV class distinguish between different small exotic smooth \mathbb{R}^4 's grouped in the so called radial family [45]. It was subsequently proposed to consider a special limit of (supersymmetric) gauge theories formulated on exotic \mathbb{R}^4 from the radial family, where geometric data of the foliations become dominant. In the following we call it the *foliated topological limit* (FTL) which is not purely topological but rather contains expressions derived from the gauge theory depend additionally on the characteristics of the foliations (and this dependence is dominant). In this way the gravitational corrections to Seiberg-Witten (SW) theory on the standard \mathbb{R}^4 are derived from the foliated topological limit of Seiberg-Witten theory formulated on exotic \mathbb{R}^4 . The driving principle is the quasi-modularity of the expressions: the universal class of every codimension-one foliation acts via multiplying by the quasi-modular Eisenstein 2-nd series, E_2 , thus the characteristics of the foliations are given by the polynomials of E_2 with modular coefficients. The same kind of expressions can be seen among the gravitational corrections to SW theory derived from superstring theory. Some of them appear as instantons corrections within the instanton calculus of Nekrasov. In general, the dependence on E_2 might indicate the role played by the codimension-one foliations.

Here we are looking for the corresponding foliated topological limit of general relativity on the exotic \mathbb{R}^4 . As usual, the problem is our highly limited knowledge of exotic 4-metrics on \mathbb{R}^4 [46] and thus we are not able to find any solution with explicit exotic metrics. However, quasi-modularity indicates the possible role of Bianchi IX geometry \mathcal{M}_3 as appearing in $\mathcal{M}_3 \times \mathbb{R}$. It is precisely the gravitational instanton where \mathcal{M}_3 is the fully anisotropic squashed 3-sphere. On the other hand $S^3 \times \mathbb{R}$ is the topological end of \mathbb{R}^4 (and the smooth end of the standard \mathbb{R}^4). In our case of GR on an exotic \mathbb{R}^4 , we find that a suitable 'geometric' end is usually based on fully anisotropic squashed $\text{sq}S^3$, i.e. $\mathcal{M}_3 \times \mathbb{R}$. One reason for this result is the metric on $\text{sq}S^3 \times \mathbb{R}$ written as function of the quasi-modular series E_2 . In contrast, the metrics on the fully symmetric $S^3 \times \mathbb{R}$ and on partially asymmetric spaces do not depend on E_2 . Moreover, quasi-modularity is expected in the special foliated topological limit of GR on exotic \mathbb{R}^4 's in the radial

family. In this way foliations and gravitational instantons meet at the semi-classical approximation to QG. However, this gravitational instanton at the end of exotic \mathbb{R}^4 is not presumably the part of the instanton solution on this exotic 4-space. Rather, the instanton $\mathcal{M}_3 \times \mathbb{R}$ gives the dominant contribution to the path integral from exotic \mathbb{R}^4 in the foliated topological limit of GR. Recall that the technique of suitable 'ends' replacing the standard one, was applied in the case of exotic \mathbb{R}_k^4 , $k = 1, 2, 3, \dots$ (from the fixed radial family). These were 'algebraic' ends, i.e. $S_k^3 \times \mathbb{R}$ which became relevant at the quantum regime of the theory on the exotics. Then, the derivation of various results from string theory was possible [8–12]. Even though, quasi-modularity characterizes foliations and exotic 4-spaces on more classical level, these expressions are derived from characters of certain superconformal algebras [13].

The connection with gravitational instantons, we believe, is fundamental for 4-d QG and allows for the inclusion of effects of exotic \mathbb{R}^4 s into the path integral. Moreover, one finds again a surprising connection of exotic 4-geometry with low energy interacting magnetic monopoles. The point is that $\mathcal{M}_3 \times \mathbb{R}$ is precisely the geometry of the moduli space of $k = 2$ low-energy BPS magnetic monopoles. However, given the BPS condition one has charges determined by mass or energy. This kind of gravitational sources can be found due to the change of the exotic smoothness on \mathbb{R}^4 to the standard one. This idea is the explanation of the coincidence between magnetic charge of Polyakov-'t Hooft monopoles and exotic 4-smoothness from the fixed radial family observed for the limiting case of Dirac monopoles in [14]. We close the paper with a discussion of the potential physical significance of monopoles, hence Higgs field, and the appearance of a massive boson A , as generated from the exotic 4-geometry on \mathbb{R}^4 .

II. QUASI-MODULARITY FROM EXOTIC \mathbb{R}^4 S FROM THE RADIAL FAMILY

The first important step of our construction is the relation of an exotic \mathbb{R}^4 to codimension-1 foliations of S^3 . To this end one fixes a radial family of small exotic smooth \mathbb{R}^4 . Let it be the Freedman-DeMichelis radial family where our exotic \mathbb{R}^4 s, e , belong to. *Every (small) exotic \mathbb{R}^4 is denoted by e in the following.* However, beginning from the conjecture in Sec. V, e denotes particular those exotic \mathbb{R}^4 s from the radial family whose ends are *modeled on AH instanton*. Recall that every small exotic \mathbb{R}^4 belongs to some radial family of such structures.

The radial family of exotic \mathbb{R}^4 s, R_t , was discovered and described firstly by Freedman and DeMichelis [7] and is the main tool for establishing many relative results about exotic \mathbb{R}^4 s. This radial family consists of a continuum of non-diffeomorphic smooth small \mathbb{R}^4 s which are defined on open subsets of standard \mathbb{R}^4 and are labeled by a single radius $t \in \text{CS}$ where CS is the standard Cantor set, such that $t_1, t_2 \in \text{CS}$ and $t_1 < t_2$ then $\mathbb{R}_{t_2}^4 \subset \mathbb{R}_{t_1}^4$. It was first proved in [14] (but see also [10, 12]) that:

Theorem 1. *Let us consider a radial family R_t of small exotic \mathbb{R}_t^4 with radii ρ , where $t = 1 - \frac{1}{\rho}$, $\rho \in \text{CS} \subset [0, 1]$, induced from the non-product h -cobordism W between M and M_0 . Then, the radial family R_t determines a family of codimension-one foliations of S^3 with Godbillon-Vey invariant ρ^2 . Furthermore, given two exotic spaces R_t and R_s , homeomorphic but non-diffeomorphic to each other (and so $t \neq s$), the two corresponding codimension-one foliations of S^3 are non-cobordant to each other.*

The following important, though direct, consequence holds true:

Corollary 1. *Every class in $H^3(S^3, \mathbb{R})$ induces a small exotic \mathbb{R}^4 where S^3 lies at the boundary of some compact subset of \mathbb{R}^4 .*

Next, given a codimension-1 foliation of a manifold N , with $\text{GV} \neq 0$, one assigns a universal GV class to it. This construction is due to Connes-Moscovici: then the universal GV class is represented by the cocycle in the cyclic cohomologies of the special universal Hopf algebra \mathcal{H}_1 . A natural action of \mathcal{H}_1 on the crossed product $\mathcal{M} \rtimes \text{GL}^+(2, \mathbb{Q})$, of the ring of modular forms \mathcal{M} and the matrix group $\text{GL}^+(2, \mathbb{Q})$, is determined. Thus, given the Theorem 1 the following crucial result emerges [13]:

Theorem 2 (Th. 6.6, [13]). *For the DeMichelis-Freedman radial family of exotic \mathbb{R}^4 's the exotic smoothness structures of the members determine a non-holomorphic canonical deformation of the modular functions from \mathcal{M} . This deformation is determined by the action of the Hopf algebra \mathcal{H}_1 on the crossed product $\mathcal{M} \rtimes \text{GL}^+(2, \mathbb{Q})$ where the GV class of the codimension-one foliations of Σ is interpreted as the cyclic cocycle of the \mathcal{H}_1 .*

From the Connes-Moscovici construction follows that the universal GV class acts via multiplication by E_2 . In that way monomials of Eisenstein series $f \cdot E_2$, and subsequently polynomials, with modular coefficients appear. The appearance of these expressions indicates that the special, half topological half geometric, limit of a theory is presumably approached and, in dimension 4, could be related, under certain conditions, with exotic 4-smoothness on \mathbb{R}^4 .

III. FOLIATED TOPOLOGICAL LIMIT OF GR ON e

Let us consider a smooth connected 4-manifold M , compact or not, and some closed 3-d smooth manifold N such that $N \subset \tilde{M}$ is a smooth submanifold. Let us suppose that there exists another smooth structure on M , \tilde{M} , and that now, $N \subset \tilde{M}$ is only a topological submanifold. Let us consider the codimension-1 foliations \mathcal{F}_N of N , with non-zero Godbillon-Vey invariants, $\text{GV}_N \in H^3(N, \mathbb{R})$. The solution of a sourceless Einstein equations (EE) on M determines a metric g_M on M (a Ricci-flat metric). Suppose that the smooth structure \tilde{M} is related with the GV class of the foliations \mathcal{F}_N s in an invariant manner which means that \tilde{M} can be determined from $\text{GV}(\mathcal{F}_N) \neq 0$. We define the foliated topological limit of GR in this case, as:

Definition 1. *The foliated topological limit (FTL) of GR on exotic \tilde{M} is given by:*

- *i. a solution of EE on \tilde{M} - a smooth metric \tilde{g} on \tilde{M} such that*
- *ii. \tilde{g} is the function of Eisenstein 2-nd quasi-modular series and \tilde{g} is derivable from the quasi-modular properties of E_2 .*

Let us recall a similar construction in the case of supersymmetric Yang-Mills theory on exotic \mathbb{R}^4 [15]. For the low energy effective theory to the $\mathcal{N} = 2$ YM, i.e. Seiberg-Witten theory on exotic \mathbb{R}^4 , the FTL of the theory was defined similarly, namely, as the regime where the existing corrections to the Langrangian on flat standard \mathbb{R}^4 are polynomials in the Eisenstein 2-nd series with modular coefficients. The main point in deriving these results was the existence of a single prepotential for SW theory so that the corrections were basically the corrections to the prepotential. However, it was obtained for $\mathcal{N} = 2$ supersymmetry which guarantees the existence of the prepotential at all. In our case of GR we do not make any use of supersymmetry and try to obtain a similar dependence on E_2 though directly at the level of metric. Since, we have the limited possibility of dealing with explicit exotic metrics on \mathbb{R}^4 , the FTL defined above serves as a general heuristic rule for grasping the dependence of GR on the foliations which determine the fake \mathbb{R}^4 (see, Sec. II).

Before going on, let us comment on the general meaning of the FTL limit which also motivates our next steps. The FTL can be understood as a kind of geometry/geometry duality in field theory and gravity. Namely, given a background manifold M the change of smoothness induces some relevant changes of the geometry. But now the new geometry relies on the foliations of N because the exotic smoothness of M depends on it. Moreover, the new geometry of exotic \tilde{M} places itself in the non-perturbative regime of the theories, hence, the connection with monopoles and instantons is rather natural. In FTL one searches for a formulation of the theories on \tilde{M} which would recover the characteristics of the foliations as the parts of the Lagrangians or the correlation functions. In the case of classical field theory, like GR, one recovers solutions of the theory as characteristics of the foliations. This is analogous to taking a 'topological' twist in some (supersymmetric) QFTs, or in theories of gravity. However, in our case of FTL, the characteristics respects rather the inherently non-commutative geometry of the foliations. In this way quantumness is already written in the classical solutions of the theory and is connected with a dual 4-geometry. Note that the geometry of instantons has also classical and as well quantum meaning in semiclassical and Euclidean QG. Searching for gravitational instanton-like geometries as assigned to dual exotic 4-geometry seems legitimate from that point of view.

IV. GRAVITATIONAL INSTANTON FROM QUASI-MODULARITY

Our next task is to find 4-d metrics which would be generated from quasi-modular expressions, in order to fulfill the requirements of Def. 1. Thus, let us begin with $y(z) = i\pi E_2(z)$, $z \in \mathbb{C}$. One additionally derives: $y'(z) = \frac{i\pi}{6}(E_2^2 - E_4)$ and $y''(z) = -\frac{i\pi^3}{18}(E_2^3 - 3E_2E_4 + 2E_6)$ where E_2, E_4, E_6 are Eisenstein series [16]. With $\vartheta_2, \vartheta_3, \vartheta_4$ the Jacobi theta-functions [16], let us introduce new variables $\omega^1, \omega^2, \omega^3$ as:

$$\begin{aligned}\omega^1 &= \frac{\pi}{6i}(E_2 - \vartheta_2^4 - \vartheta_3^4) \\ \omega^2 &= \frac{\pi}{6i}(E_2 - \vartheta_3^4 - \vartheta_4^4) \\ \omega^3 &= \frac{\pi}{6i}(E_2 + \vartheta_2^4 - \vartheta_4^4),\end{aligned}\tag{1}$$

so that $y = -2(\omega^1 + \omega^2 + \omega^3)$ and $y' = 2(\omega^1\omega^2 + \omega^2\omega^3 + \omega^3\omega^1)$, $y'' = -12\omega^1\omega^2\omega^3$ and the Jacobian of the coordinate change reads: $J = (\omega^1 - \omega^2)(\omega^2 - \omega^3)(\omega^3 - \omega^1)$, which is non-zero for $\omega^1 \neq \omega^2 \neq \omega^3$.

Now, let us consider the so called Darboux-Halphen (DH) system of differential equations on $\gamma^i(z)$, $z \in \mathbb{C}$, $i = 1, 2, 3$:

$$\begin{aligned}\frac{d\gamma^1}{dz} &= \gamma^2\gamma^3 - \gamma^1(\gamma^2 + \gamma^3) \\ \frac{d\gamma^2}{dz} &= \gamma^2\gamma^3 - \gamma^1(\gamma^2 + \gamma^3) \\ \frac{d\gamma^3}{dz} &= \gamma^2\gamma^3 - \gamma^1(\gamma^2 + \gamma^3) .\end{aligned}\tag{2}$$

It is known that unlike the other cases, the fully anisotropic one, i.e. $\gamma^1 \neq \gamma^2 \neq \gamma^3$, does not allow for algebraic integrals. The solutions are, however, expressible in terms of quasi-modular series E_2 . Namely, given the base for quasi-modular forms of weight 2, i.e. \mathcal{E}^i , $i = 1, 2, 3$, the fully anisotropic solutions of DH system (2), read:

$$\gamma^i(z) = -\frac{1}{2} \frac{d}{dz} \log \mathcal{E}^i(z), i = 1, 2, 3 .\tag{3}$$

Defining $\lambda = \frac{\gamma^1 - \gamma^3}{\gamma^1 - \gamma^2}$ one expresses \mathcal{E}^i , $i = 1, 2, 3$ as:

$$\mathcal{E}^1 = \frac{1}{\lambda} \frac{d\lambda}{dz}, \mathcal{E}^2 = \frac{1}{\lambda - 1} \frac{d\lambda}{dz}, \mathcal{E}^3 = \frac{1}{\lambda(\lambda - 1)} \frac{d\lambda}{dz} .\tag{4}$$

Now, the choice of $\lambda = \frac{\vartheta_2^4}{\vartheta_3^4}$ and calculating \mathcal{E}^i result in:

$$\mathcal{E}^1 = i\pi\vartheta_4^4, \mathcal{E}^2 = -i\pi\vartheta_2^4, \mathcal{E}^3 = -i\pi\vartheta_3^4 .$$

Equivalently, one derives γ^i from (3) as:

$$\gamma^1 = \frac{\pi}{6i}(E_2 - \vartheta_2^4 - \vartheta_3^4), \gamma^2 = \frac{\pi}{6i}(E_2 - \vartheta_3^4 - \vartheta_4^4), \gamma^3 = \frac{\pi}{6i}(E_2 + \vartheta_2^4 - \vartheta_4^4) .\tag{5}$$

However, these are precisely the expressions for ω^i as in (1), i.e. $\omega^i = \gamma^i$, $i = 1, 2, 3$. Starting from $i\pi E_2$ we arrive at the solutions of (2). Now we will see that solving Eqs. 2 is nothing but finding certain 4-d metric which is a gravitational instanton. First, we are going to show that the requirement for self-duality in the GR equations on certain 4-manifold is recapitulated precisely in the system (2) of DH equations. Let \mathcal{M}_3 be 3-space of Bianchi IX type, i.e. the Killing vectors ξ_i , $i = 1, 2, 3$ form the $SU(2)$ algebra:

$$[\xi_i, \xi_j] = c^i_{jk} \xi_k$$

and the Maurer-Cartan 1-forms, σ^i , $i = 1, 2, 3$ for this Bianchi group fulfill:

$$d\sigma^i = \frac{1}{2} c^i_{jk} \sigma^j \wedge \sigma^k .$$

In terms of the Euler angles, $0 \leq \alpha \leq \pi, 0 \leq \beta \leq 2\pi, 0 \leq \psi \leq 4\pi$, these 1-forms read:

$$\begin{aligned}\sigma^1 &= \sin \alpha \sin \psi d\beta + \cos \psi d\alpha \\ \sigma^2 &= \sin \alpha \cos \psi d\beta - \sin \psi d\alpha \\ \sigma^3 &= \cos \alpha d\beta + d\psi .\end{aligned}\tag{6}$$

A metric $ds_3^2 = g_{ij} \sigma^i \sigma^j$ on \mathcal{M}_3 can be always diagonalized. Let us take our 4-d manifold $\mathcal{M}_3 \times \mathbb{R}$. General metric on it reads:

$$ds^2 = dt'^2 + g_{ij}(t') \sigma^i \sigma^j .$$

However, this metric can be always diagonalized too, and with the use of arbitrary functions $\Theta^i(t')$, $i = 1, 2, 3$, such that $dt' = \sqrt{\Theta^1 \Theta^2 \Theta^3} dt$, a general metric on $\mathcal{M}_3 \times \mathbb{R}$ becomes:

$$ds^2 = \Theta^1 \Theta^2 \Theta^3 dt^2 + \frac{\Theta^2 \Theta^3}{\Theta^1} (\sigma^1)^2 + \frac{\Theta^1 \Theta^3}{\Theta^2} (\sigma^2)^2 + \frac{\Theta^1 \Theta^2}{\Theta^3} (\sigma^3)^2 .\tag{7}$$

Now, as usual in the 4-d Cartan formalism, the basis of 1-forms on a 4-manifold M is given by $\{\theta^a\}$. Then the corresponding connection 1-forms $\omega^a_b = \Gamma^a_{bc} \theta^c$, $a, b, c = 1, 2, 3, 4$, the curvature 2-form reads:

$$\mathcal{R}^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b = \frac{1}{2} R^a_{bcd} \theta^c \wedge \theta^d\tag{8}$$

where R_{bcd}^a are the components of the Riemann tensor. Let us now decompose the Riemann curvature \mathcal{R}_{ab} and the connection ω_{ab} into self-dual and anti-self-dual parts with respect to the duality defined by:

$$\begin{aligned}\tilde{\omega}_b^a &= \frac{1}{2}\varepsilon_{bc}^a \omega_d^c \\ \tilde{\mathcal{R}}_b^a &= \frac{1}{2}\varepsilon_{bc}^a \mathcal{R}_d^c\end{aligned}\quad (9)$$

and with respect to the factorization $SO(4)$ into $SO(3) \times SO(3)$. Then, writing for $i, j, k = 1, 2, 3$:

$$\begin{aligned}s_i &= \frac{1}{2}(\omega_{0i} + \frac{1}{2}\varepsilon_{ijk}\omega^{jk}), & a_i &= \frac{1}{2}(\omega_{0i} - \frac{1}{2}\varepsilon_{ijk}\omega^{jk}), \\ S_i &= \frac{1}{2}(\mathcal{R}_{0i} + \frac{1}{2}\varepsilon_{ijk}\mathcal{R}^{jk}) & A_i &= \frac{1}{2}(\mathcal{R}_{0i} - \frac{1}{2}\varepsilon_{ijk}\mathcal{R}^{jk}),\end{aligned}\quad (10)$$

one has $s_i = \tilde{s}_i$, $S_i = \tilde{S}_i$ and $a_i = -\tilde{a}_i$, $A_i = -\tilde{A}_i$ under the duality (9). Moreover, \mathcal{R}_b^a in (8) is given by:

$$\begin{aligned}S_i &= ds_i - \varepsilon_{ijk}s^j \wedge s^k \\ A_i &= da_i + \varepsilon_{ijk}a^j \wedge a^k,\end{aligned}\quad (11)$$

so that s_i, S_i are self-dual components of ω and \mathcal{R}_b^a , while a_i, A_i are anti-self-dual their components. In this way the self-duality equations of GR read:

$$A_i = da_i + \varepsilon_{ijk}a^j \wedge a^k = 0. \quad (12)$$

In terms of the diagonalizing functions $\Theta^i, i = 1, 2, 3$ for our case of $M = \mathcal{M}_3 \times \mathbb{R}$, we have:

$$a_i = \frac{1}{4\sqrt{\Theta^1\Theta^2\Theta^3}} \left[\frac{1}{\Theta^i} \left(\frac{d\Theta^i}{dt} - \Theta^j\Theta^k \right) - \frac{1}{\Theta^j} \left(\frac{d\Theta^j}{dt} - \Theta^k\Theta^i \right) - \frac{1}{\Theta^k} \left(\frac{d\Theta^k}{dt} - \Theta^i\Theta^j \right) \right]. \quad (13)$$

One solves (12) for non-zero a_i yielding $a_i = \frac{1}{2}\delta_{ij}\sigma^j$, which together with (13) gives the following system of equations:

$$\begin{aligned}\frac{d\Theta^1}{dt} &= \Theta^2\Theta^3 - \Theta^1(\Theta^2 + \Theta^3) \\ \frac{d\Theta^2}{dt} &= \Theta^3\Theta^1 - \Theta^2(\Theta^3 + \Theta^1) \\ \frac{d\Theta^3}{dt} &= \Theta^1\Theta^2 - \Theta^3(\Theta^1 + \Theta^2)\end{aligned}\quad (14)$$

which is nothing but the real line version of the Darboux-Halphen equations (2). To retrieve real solutions $\Theta^i(t)$ from the complex ones $\gamma^i(z)$ as in (3), one simply takes:

$$\Theta^k(t) = i\gamma^k(it) = -\frac{1}{2}\frac{d}{dt}\log\mathcal{E}^k(it), k = 1, 2, 3. \quad (15)$$

In this way we have a metric on $\mathcal{M}_3 \times \mathbb{R}$ as in (7) whose coefficients depend on E_2 as in (1) and which were, in fact, determined from the quasi-modularity of E_2 . To complete the story we need to show that this metric is the gravitational instanton. But, its self-duality was imposed by (12) and (13) and this condition is in fact realized in the DH eqs. (14). This solution has a long story and was also obtained by Atiyah and Hitchin in the context of monopoles [17, 18], so it is called Atiyah-Hitchin (AH) instanton.

V. GRAVITATIONAL INSTANTON AND MAGNETIC MONOPOLES FROM EXOTIC \mathbb{R}^4

In this section we try to go further and relate the gravitational instanton on $\mathcal{M}_3 \times \mathbb{R}$, obtained in the previous section, with the 'effective' geometry of the end of the exotic \mathbb{R}^4 .

$SU(2)$ YM theory can be formulated on every smooth, compact or not, 4-manifold. Moreover, it is known that the YM $SU(2)$ instanton exists on every compact oriented 4-d smooth manifold [19]. It was believed that it exists also on open oriented 4-d manifolds. However, the counterexample was constructed in [20] showing that the infinite connected sum of the complex projective space, $(CP^2)^{\# \mathbb{Z}}$, does not allow for any (non-flat) YM instanton solution. There is also a close relation between YM and gravitational instantons in the sense that a large class of YM instantons is solved at the same time by a gravitational instanton [21].

Thus, currently, we can not decide definitely whether gravitational instantons exist on exotic \mathbb{R}^4 s. Instead, we propose to associate one with these curved open 4-manifolds such that, in some limit, these instantons have dominant contributions to the path integral. So we perform two deformations, one when the standard smooth structure on \mathbb{R}^4 is changed to e , and the second within GR on e by taking FTL. As usual we fix the radial family which the exotic e

belongs to. As so, the (cobordant classes of) codimension-1 foliations of S^3 with non-zero GV, label exotics e 's from the family. In fact, given $GV \neq 0$ one determines the (cobordant classes of) codimension-1 foliations of S^3 from the radial family. Taking the FTL limit of a theory on e is similar to the topological invariants of the manifolds obtained as the correlation functions in the topological twist of a quantum field theory. Here we have a rather quasi-topological twist of a theory where the expressions depend on E_2 hence on the class of the foliation. So, the quasi-modular character of metrics in the FTL limit of GR on e is expected. This is also an indication that the instanton, given in (7) and (1), might be relevant here.

Let us take a closer look at the geometry of this instanton. The case of the DH equations makes use of $\Theta^1 \neq \Theta^2 \neq \Theta^3$. The partially anisotropic case, as e.g., $\Theta^1 = \Theta^2$, and the fully symmetric one, $\Theta^1 = \Theta^2 = \Theta^3$, do not require any reference to quasi-modular expressions. These cases are solved by algebraic means and are related with other gravitational instantons, like Eguchi-Hanson or Taub-NUT ones. However, the geometry of the fully symmetric case allows for isometries given by left-invariant killing vectors (6) creating one copy of $SU(2)$, and by the other 3 right invariant killing vectors spanning the other $SU(2)$. The isometries are, thus, $SU(2) \times SU(2)$, the spin group $Spin(4)$ of the Euclidean space. 4-d geometry resulting from these symmetries is $S^3 \times \mathbb{R}$. This is the topological, and smooth, end of the standard \mathbb{R}^4 . The partially anisotropic case reduces the isometries to $SU(2)$ left invariant, as in (6), and breaks the right invariant $SU(2)$ to $U(1)$, so the result is $SU(2) \times U(1)$. The geometry of this case is based on the partially anisotropic (partially squashed) sphere i.e., $psqS^3 \times \mathbb{R}$.

Finally, the fully anisotropic, and quasi-modular, case is given by the $SU(2)$ isometries, and its geometry is based on the fully anisotropic 3-sphere, i.e. $sqS^3 \times \mathbb{R}$. This 'subtle' squashing of S^3 is the reason that one can not solve the self-duality equations algebraically, and quasi-modularity has to intervene. We have to remember that the 3-d metrics of the 3-spheres depend on the t component and this t -dependence of the metrics is quasi-modular. Thus, the deformation of S^3 (anisotropic squashing) and the requirement of self-duality on $sqS^3 \times \mathbb{R}$ automatically gives rise to the quasi-modular dependence on t . Without squashing the self-duality alone does not produce any quasi-modular expressions.

We refer to the squashing of a 3-sphere as associated to the change of the smoothness structure on \mathbb{R}^4 . However, the effects due to the change of smoothness are on the topological counterpart of S^3 , since it and the squashed sqS^3 are not smoothly embedded into exotic \mathbb{R}^4 . Moreover, the resulting embedding of S^3 should be based on some wild embeddings. However, the effects are understood in the sense of dominant contributions to the path integral. Any direct calculation on e is not possible at present. We are rather approaching qualitatively the change of the standard smoothness of \mathbb{R}^4 into an exotic e from the fixed radial family. The smoothness of e can not be localized into a smooth 4-disk, so the exotic geometry deforms the standard end $S^3 \times \mathbb{R}$. The exotic end of e is not a smooth $S^3 \times \mathbb{R}$ nor it is a smooth $sqS^3 \times \mathbb{R}$. We do not know how precisely the geometry of this deformation looks like, however in the FTL of GR on e the quasi-modularity should be dominant. Moreover, from the point of view of the path integral this should be assigned to the solution minimizing the action. Both conditions are met on the 'squashed end' $sqS^3 \times \mathbb{R}$ of e which is the case of the fully anisotropic AH instanton. More arguments of purely mathematical character will be presented separately, here we rather focus on possible physics behind it. So, let us formulate this relation as the following conjecture:

Conjecture 1. *There exists an exotic \mathbb{R}^4 , e , belonging to the DeMichelis-Freedman radial family such that:*

The foliated topological limit of general relativity on e is represented by the geometric end given by the Atiyah-Hitchin instanton in the sense that the contribution from this gravitational instanton approaches the dominant contribution from e into the gravitational Euclidean path integral.

Note that when smoothness of \mathbb{R}^4 is fixed to the standard one, the foliations have vanishing GV class, hence the whole quasimodularity disappears (from the Connes-Moscovici construction). This corresponds to $sqS^3 \times \mathbb{R} \rightarrow S^3 \times \mathbb{R}$. However, beginning with the standard \mathbb{R}^4 and its standard end $S^3 \times \mathbb{R}$ and searching for its deformation such that the dependence on E_2 becomes apparent, one arrives uniquely at the stable geometry of the fully anisotropic S^3 , i.e. $\mathcal{M}_3 \times \mathbb{R}$.

Now let us comment on the contributions to the path integral of Euclidean quantum gravity (EQG). Usually, the calculation of this path integral is the integration over different classes of positive definite metrics $g_{\mu\nu}$ on e , i.e.:

$$Z_e = \int D[g_{\mu\nu}] e^{-S[g_{\mu\nu}, e]}$$

where the action can read:

$$S[g_{\mu\nu}] = -\frac{1}{16\pi} \int_e (R - 2\Lambda) \sqrt{g} d^4x - \frac{1}{8\pi} \int_{\partial e} K \sqrt{h} d^3x = -\frac{1}{16\pi} \int_e (R - 2\Lambda) \sqrt{g} d^4x.$$

Given the decomposition of the standard \mathbb{R}^4 as $\mathbb{R}^4 = D^4 \cup S^3 \times \mathbb{R}$ the path integral on the standard \mathbb{R}^4 reads

$$Z_{\mathbb{R}^4} = \int D[h] \left(\int D[g'] e^{-S[g', h, D^4]} \int D[g''] e^{-S[g'', h, \mathbb{R}^4 \setminus D^4]} \right)$$

which is not exotic-smooth valid, since $(S^3, h) = (\partial D^4, h)$ is not smoothly embedded in e any longer, and h is a 3-metric on S^3 . Now, we change the smooth structure to e . The decomposition above do not respects the exotic smoothness structure. However, in the exotic e case, we still can speak about the (exotic) end, $\text{end}(e)$, such that e is decomposed as $N^4 \cup \text{end}(e)$. In the FTL of GR on e the conjecture states that Z_e^{FTL} is approximated by the $\int D^{FTL}[g]e^{-S[g, \mathcal{M}_3 \times \mathbb{R}]}$ in the sense that the leading contribution from the $\text{end}(e)$ in FTL of GR on e , is calculated *locally* from the AH instanton in the sense given below.

Let $g_{\mu\nu}^0$ be a flat metric on \mathbb{R}^4 with its restriction $g_{\mu\nu}^0|_{S^3 \times \mathbb{R}}$ to the standard end $S^3 \times \mathbb{R}$. Next, we are changing the smooth structure on \mathbb{R}^4 to e . A non-flat metric on e is $g_{\mu\nu}^e$ which is non smooth in the standard \mathbb{R}^4 . Working in the standard smoothness there is the difference $\tilde{\Phi}_{\mu\nu}$ (non-standard smooth) between the metrics, such that $g_{\mu\nu}^e = g_{\mu\nu}^0 + \tilde{\Phi}_{\mu\nu}$ and $\tilde{\Phi}_{\mu\nu}$ is still continuous. Let us approximate it by the sequence of standard smooth functions $\Phi_{\mu\nu}^\epsilon$ converging to $\tilde{\Phi}_{\mu\nu}$. Now $g_{\mu\nu}^\epsilon = g_{\mu\nu}^0 + \Phi_{\mu\nu}^\epsilon$ is standard smooth. However, these $\Phi_{\mu\nu}^\epsilon$, 'sufficiently well' (truly) approximated exotic non-smooth corrections can not be small perturbations everywhere. They are rather global effects of the exotic smooth e .

In the semi-classical QG approximation we consider small $\phi_{\mu\nu}^\epsilon$ corrections as (quantum) perturbations which can be taken as c -number fields. Then, one can expand the path integral over the base geometry. This base geometry is the saddle-point of the classical action and is given by the complete, finite-action, metric, i.e. gravitational instanton. For the case of the AH gravitational instanton, we have a local expansion around it:

$$Z^{\text{std}}[g_{\mu\nu}^{AH}, \mathcal{M}_3 \times \mathbb{R}] \approx e^{-S(g_{\mu\nu}^{AH})} \int D\phi^\dagger D\phi \exp \left[-\frac{1}{2} \int \sqrt{g} d\tau d^3x \phi^\dagger \mathcal{A}(g^{AH}) \phi \right] = \exp \left\{ -S(g^{AH}) - \frac{1}{2} \ln \det[\mathcal{A}(g^{AH})] \right\}$$

where \mathcal{A} is the quantum operator representing the corrections. The claim in the conjecture states that the above expression is the correct approximation for the path integral for GR on e in FTL. In fact, this is the approximation for the evaluation of the path integral on the exotic end of e in FTL, i.e.

$$Z^e[g^{FTL}, \text{end}(e)] \approx Z^{\text{std}}[g_{\mu\nu}^{AH}, \mathcal{M}_3 \times \mathbb{R}]. \quad (16)$$

Certainly, we do not know an explicit shape of the operator \mathcal{A} but we state its existence and that \mathcal{A} depends on the particularities of FTL of GR on e . Note that one can consider also other fields on e when path integral is approached. This is due to the fact that the change of the smoothness modifies every smooth object on \mathbb{R}^4 .

One can say that the functional measure in FTL respects the universal class of the codimension-1 foliations of S^3 in a sense that the resulting expression depends also on the class of the foliation. Suppose that FTL is attained for GR on e . Then, the relevant metrics in this limit depend on E_2 and these metrics contribute to the path integral. The conjecture states that Z_e^{FTL} is approximated by the contribution from the AH instanton, even though it is not any exotic geometry at the end of the exotic e . As we will see the AH geometry rather emerges as a kind of 'symmetry breaking' in exotic geometry by a scalar field ϕ . This will enforce the $U(1)$ direction in the $SU(2)$ isospace and the quasi-modular direction (t) in 4-space.

To grasp contributions of other exotic \mathbb{R}^4 s from this radial family into the path integral, one should additionally respect the dependence on the particularities of the solutions of the DH system and/or on the non-universal GV classes of the foliations distinguishing between the members of the radial family. The attempt to calculate directly the gravitational path integral on exotic \mathbb{R}^4 s from the radial family, was performed recently [22]. It would be interesting to compare both approaches and check whether the results agree in the case of a fixed radial family, where dependence on the GV class is expected. We will address this possibility separately.

Thus, when we deal with gravity path integral in the FTL of GR on e , the dominant contributions are derivable from the AH instanton metric, and we, then, say that e has *geometric end modeled on the AH instanton*.

The profound physical consequences can be drawn from the existence of this exotic \mathbb{R}^4 with the geometric end modeled on the Atiyah-Hitchin instanton in the FTL limit of GR. The relation with magnetic monopoles is one consequence. Let us recall the following result which was obtained originally in [14]:

Some small, exotic smooth structures on \mathbb{R}^4 can act as sources of magnetic field, i.e. monopoles, in spacetime. Electric charge in spacetime has to be quantized, provided some region has this small exotic smoothness.

This result was obtained due to the algebraic coincidence of universal classes (cobordism) of the foliations and of Dirac monopoles. Here we can go much further. We will show that the exotic e as in the conjecture, is related analytically with magnetic monopoles of Polyakov-'t Hooft type, and this e serves as the example of the exotic 4-region. Moreover, there is the mechanism for the geometric generation of the scalar Higgs field in 4-spacetime.

How is it possible that 4-geometry generates any monopole configuration at all? Such possibility is encoded in the BPS condition. Let us consider the non-flat geometry on exotic e . According to early results on exotic \mathbb{R}^4 s [23–26] suppose that e generates a gravitational source $T_{\mu\nu} \neq 0$ which becomes apparent when the exotic smoothness of e is changed into the standard one. Thus, e generates the non-zero source $T_{\mu\nu}$, though Einstein equations on e would be still sourceless in the exotic structure. The non-trivial source $T_{\mu\nu}$ is, as usual, connected with some distribution of matter and energy in spacetime. Let us assume additionally that the matter in the static limit is related with

particles (classical or quantum) with non-zero masses m . According to the above discussion it is natural to consider m as the mass of magnetic monopoles. However, the BPS condition relates the charges (of dyons or monopoles) with its masses. So, in the standard smooth structure of \mathbb{R}^4 the particles with mass m bear the monopole (dyon) charge. Moreover, the BPS condition is crucial for determining the geometry $\text{sq}S^3 \times \mathbb{R}$ as the geometry of the moduli space with $k = 2$. This geometry, in turn, is derivable from the exotic e due to quasi-modularity and appears as the crucial player in evaluation of the gravitational path integral on e . Note that, one can solve analytically the Yang-Mills-Higgs equations only when the BPS condition is assumed. Let us approach this in more detail.

Given the $SU(2)$ principal bundle on Minkowski space M^4 , let $A = A_\mu = A_\mu^a T^a$ be the connection on the bundle with the field strength $F = F_{\mu\nu} = F_{\mu\nu}^a T^a$ and the standard normalization of the generators of the gauge group is $\text{Tr}(T^a T^b) = \frac{1}{2} \delta_{ab}$. For $SU(2)$, $T^a = \frac{\sigma^a}{2}$. The Lie algebra, as usual, is defined via $[T^a, T^b] = i \varepsilon_{abc} T^c$.

For the covariant derivative $D = D_A = d + A$ in this principal $SU(2)$ bundle, the Yang-Mills equations read:

$$D_A F = 0, \quad D_A \star F = 0$$

where as usual \star is the Hodge \star -operator.

The Yang-Mills boson, as well the photon in the Maxwell equations, has to be massless due to the gauge symmetry. To introduce a gauge invariant mass terms into the theory one introduces the Higgs field $\phi = \phi^a T^a$, $a = 1, 2, 3$ (summation), so that the action density now becomes:

$$L = -\frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \text{Tr} D_\mu \phi D^\mu \phi - V(\phi) = -\frac{1}{4} \text{Tr} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} \text{Tr} (D^\mu \phi^a) (D_\mu \phi^a) - \tilde{V}(\phi). \quad (17)$$

This is the Yang-Mills-Higgs action density. Suppose also that ϕ transforms in the adjoint of $SU(2)$, so that the potential can be chosen as $V(\phi) = -\tilde{V}(\phi) = -\lambda(|\phi|^2 - v^2)^2$ and when $v^2 \neq 0$ the $SU(2)$ symmetry is broken to $U(1)$. The action density now reads:

$$L = (F, F) + (D\phi, D\phi) + V(\phi) = -\frac{1}{4} \text{Tr} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} \text{Tr} (D^\mu \phi^a) (D_\mu \phi^a) - \lambda(|\phi|^2 - v^2)^2. \quad (18)$$

Now the mass term for the vector boson A , i.e. $m^2 f(A^2)$, is introduced as follows. The minimal energy of the configuration from (17) is attained for $V = F = D_A \phi = 0$ and the Higgs field

$$\phi^a \xrightarrow[r \rightarrow \infty]{vr^a/r} \phi_0 = \text{const}, \quad (19)$$

and in the vacuum background ϕ_0 and ϕ are related by gauge transformation, so that

$$D\phi_0 = d\phi_0 + A\phi_0 = A\phi_0,$$

which means $(D\phi_0, D\phi_0) = (A\phi_0, A\phi_0)$. Introducing it into the background action $L = (F, F) + (D\phi_0, D\phi_0)$ we obtain the quadratic in A mass term.

The vector bosons A^\pm acquire charges due to the unbroken $U(1)$ which leaves the Higgs vacuum invariant (a direction in the isospin space). So, one represents this subgroup by the rotations about the direction in the isospace given by ϕ^a and the generator of this rotation is $\frac{\phi^a T^a}{\sqrt{\phi^a \phi^a}}$ or $\frac{\phi^a T^a}{a}$ where $\phi^a \phi^a = a^2$. It is the operator of the electric charge Q at the same time. The covariant derivative now reads:

$$D_\mu = \partial_\mu + iQ A_\mu^{(e)}$$

where $A_\mu^{(e)} = \frac{1}{a} A_\mu^a \phi^a$ and $Q = e \frac{1}{a} \phi^a T^a$. $A_\mu^{(e)}$ is the projection of $SU(2)$ A_μ^a onto the direction of ϕ^a . The corresponding strength is $F_{\mu\nu}^a = F_{\mu\nu} \frac{\phi^a}{a}$, where

$$F_{\mu\nu}^a = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{a^3 e} \varepsilon_{abc} \phi^a \partial_\mu \phi^b \partial_\nu \phi^c. \quad (20)$$

The non-trivial correction to the electrodynamic field strength $F_{\mu\nu}$ as seen in (20) vanishes in the topologically trivial sector of the theory, while it is non-zero when the boundary condition are as in (19). In this case the second pair of Maxwell equations becomes $\partial^\mu \tilde{F}_{\mu\nu} = k_\nu$ where

$$k_\mu = \frac{1}{2a^3 e} \varepsilon_{\mu\nu\rho\sigma} \varepsilon_{abc} \partial^\nu \phi^a \partial^\rho \phi^b \partial^\sigma \phi^c$$

is the magnetic current absent in the pure electrodynamic case. Let us observe that the current is expressible only via the Higgs field. It is the first indication that, when the Dirac monopoles-magnetic sources were found to be corresponding to some exotic \mathbb{R}^4 s, then we found the same mechanism for the Higgs field.

Next, turn to the monopole solutions. The EOM derived from (17) are:

$$\begin{aligned} D_A F &= 0 \\ D_A \star F &= -[\phi, D_A \phi] \\ D_A \star D_A \phi &= 2\lambda\phi(|\phi|^2 - v^2) \end{aligned} \quad (21)$$

where the first equation is the Bianchi identity. The configuration (A, ϕ) solving the equations (21) is the classical configuration representing magnetic monopoles in the sense that every quantum theory of magnetic monopoles should have the above configuration of fields in the classical limit. (A, ϕ) is the configuration of fields defining the magnetic monopole. The special static regular solutions were numerically determined as:

$$\phi^a = \frac{r^a}{er^2} H(\xi), \quad A_n^a = \varepsilon_{amn} \frac{r^m}{er^2} (1 - K(\xi)), \quad A^a = 0, \quad \xi = ver,$$

H, K being some functions of ξ . These are the Polyakov-'t Hooft solutions. Even though solving the YMH equations (21) in general, on H, K , is intractable analytically, but the special limit of Bogomol'nyi-Prasad-Sommerfield (BPS) can be solved. This case is of special importance for the connection with the exotic 4-geometries on \mathbb{R}^4 . Namely the BPS limit is determined by the vanishing scalar potential $V(\phi) = 0$ and the minimum of the energy is reached by the static configurations which solves the 3-d Bogomol'nyi equation:

$$\varepsilon_{ijk} F_{ij} = \pm D_k \phi. \quad (22)$$

These BPS monopoles are determined by the explicit K and H , namely: $K(\xi) = \frac{\xi}{\sinh \xi}$, $H(\xi) = \xi \coth \xi - 1$.

The crucial property of the BPS solutions is the connection between mass and charge. The Bianchi identity and (22) gives $D_{A_n} D_{A_n} \phi^a = 0$, so that the energy of the BPS monopole reads: $E = \frac{1}{2} \int dx \partial_n \partial_n (\phi^a \phi^a) = gv$. Hence, when the dyon has electric q and magnetic g charges, the mass of such BPS state is given by the Bogomol'nyi bound:

$$M = v \sqrt{g^2 + q^2}. \quad (23)$$

The BPS solutions can be understood as the direct reduction of Euclidean 4-d YM theory into 3-dimensions. Namely, one considers pure YM equation in the Euclidean \mathbb{R}^4 with an invariance of x_4 translation, so that the connection reads $A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 + \phi dx_4$. A_i and ϕ are now $SU(2)$ -algebra valued functions on \mathbb{R}^3 and the Euclidean action is written as $L = (F, F) + (D\phi, D\phi)$ where the curvature F and the derivative D are defined with respect to the connection $A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3$. Now one sees that the YMH theory on \mathbb{R}^3 with $V(\phi) = 0$ is the dimensional reduction of a pure YM on Euclidean \mathbb{R}^4 . Note that $V(\phi) = 0$ is the BPS condition. Moreover, static monopole BPS solutions are derived from the 4-d YM self-dual ones, i.e. $F = \star F$. However, a more geometric picture is possible which is important in this paper and is attained when classical solutions are seen as point-particles in the space of solutions [18].

Given the space of the 3-d solutions (A, ϕ) modulo gauge invariance, $\mathcal{C} = \mathcal{A}/\mathcal{G}$, the tangent vector is $(\dot{A}, \dot{\phi})$ where one differentiate with respect to some parameter of the path in \mathcal{C} . Then, the (formal) Riemannian metric on \mathcal{C} :

$$|(\dot{A}, \dot{\phi})|^2 = \int_{\mathbb{R}^3} (\dot{A}, \dot{A}) + (\dot{\phi}, \dot{\phi}) \quad (24)$$

and the potential energy $U = \frac{1}{2} \int_{\mathbb{R}^3} (F, F) + (D\phi, D\phi)$ are defined. We see that the motion of a particle on \mathcal{C} with the potential U is governed by the solutions of the YMH equations with the BPS condition. Thus, one determines the region (submanifold) $M \subset \mathcal{C}$ on which U attains the minimum. To ensure that U exists the following is assumed $|F| = O(r^{-2})$, $D\phi = O(r^{-2})$, so that $|\phi| \xrightarrow{r \rightarrow \infty} \text{constant} = v^2$. So, we are taking a 3-ball B_r of radius r and integrate

$$\int_{B_r} (F, F) + (D\phi, D\phi) = \int_{B_r} (F - \star D\phi, F - \star D\phi) + 2(\star D\phi, F) \quad (25)$$

where the second term is the surface integral over S^3 , i.e. $2 \int_{S_r} (\phi, F)$, of the 2-form (ϕ, F) since the (total) differential of it reads $d(\phi, F) = (D\phi, F) = \star(\star D\phi, F)$ and the Bianchi identity holds, i.e. $DF = 0$. This is the topological term and with its particular value $v^2 = 1$, i.e. $|\phi| \xrightarrow{r \rightarrow \infty} 1$, one has:

$$\lim_{r \rightarrow \infty} \int_{S_r} (\phi, F) = \pm 4\pi k$$

where $\pm k$ can be seen as the Chern classes of the complex line bundle on S_r related with ϕ [18]. This is the magnetic charge of the solution. In this way the condition for the absolute minimum of the potential energy U is attained whenever the first term in (25) vanishes, i.e. it holds:

$$F = \star D\phi \quad (26)$$

which is the Bogomolnyi equation (22).

Let us turn to the abelian $U(1)$ electrodynamic where the electric field E is the 1-form on \mathbb{R}^3 , the magnetic field B is the 2-form and A being the electromagnetic potential (2-form), then, the field strength tensor, reads: $F = B + cdt \wedge E$ and $F = dA$. The pure Maxwell equations in the Minkowski spacetime are:

$$dF = 0, \quad d \star F = 0.$$

The Bogomolnyi equations for the abelian case become $B = \text{grad } \phi$, and taking $\phi = \frac{k}{2r}$ one finds that this is the Dirac magnetic monopole. Moreover, B is the curvature of a connection on the line bundle on $\mathbb{R}^3 \setminus \{0\}$ which are classified by $H^3(S^3, \mathbb{Z})$. The appearance of this cohomology group was the reason for the connection of the Dirac monopoles with the special smooth \mathbb{R}^4 from the radial family, namely those which are generated by the foliations with $GV = k$ [14].

The geometric perspective on the space of BPS monopoles solutions is crucial for further exploring the relation of monopoles with exotic 4-geometries. We remark that the geometry of the moduli space of BPS monopoles can be exactly described and it is given by the complete and non-singular metric. Such exact, precise and analytical results are always of exceptional value and should be considered carefully.

Let us approach the geometry of the monopole moduli space in more detail. For us the interesting case of $k = 2$ static and weakly interacting two BPS monopoles was described by Atiyah and Hitchin [18]. Following their analysis one observes that for $k = 1$ the center of the single BPS static monopole is fixed to a single point. So, the reduced moduli space for $k = 1$ is $M_1^0 = \text{pt.}$, whereas, taking into account the symmetries of this choice, the full moduli space M_1 reads $M_1 = \mathbb{R}^3 \times S^1$. In general the relation between the reduced and full moduli spaces can be written as $M_k^0 = \frac{M_k}{\mathbb{R}^3 \times S^1}$. For us, the interesting case is the geometry of M_2^0 . Analysis of it is rather involved and based on profound works by Donaldson, Hurtubise and Taubes among others (see, e.g. [27–29]) which were extensively used by Atiyah and Hitchin. As follows from the work of Donaldson the parameter space of monopoles is very efficiently modeled by the space of rational functions. Every k BPS monopole, as the solution of (21) with the BPS condition, determines a connection, A , on the power of the line Hopf bundle H on S^2 , i.e. $H^k \otimes H^{-k}$, which is prolonged radially to $S^2 \times \mathbb{R}^+$ by the Higgs field ϕ . Given $S^2 \times \mathbb{R}^+ = \mathbb{R}^3 \setminus \{0\}$ we have the $SU(2)$ bundle on $S^3 \setminus \{0\}$ with the $U(1)$ symmetry due to Higgs. If one takes the space of gauge equivalent monopoles of charge k , N_k , then the above M_k is fibered over N_k with fiber S^1 . A point pt. in M_k represents a monopole bundle E_k up to some asymptotic isomorphism of $E_k(\text{pt.})$ [18].

To relate this moduli space with rational functions it was proposed to consider the scattering associated with the differential operator D_A defined along the line u in E_k and acting on the sections of the bundle E_k over u :

$$D_A^u = \nabla_A^u - i\phi \quad (27)$$

where ∇_A^u is the covariant derivation on E_k along u . Thus, given the magnetic BPS monopole (A, ϕ) as the solution of the Bogomolny equation (22) we can determine the differential equation along u as (27). In fact one chooses the family of parallel lines $u(t, z), t \in \mathbb{R}, z \in \mathbb{C}$ in \mathbb{R}^3 . The equation $D_A^u s = 0$ has two independent solutions $s_0(t, z), s_1(t, z)$ whose first solution decays at $t \rightarrow \infty$ so that $s_0(t, z)t^{-k/2}e^t \rightarrow e_0$. Here e_0, e_1 are the constant sections of the asymptotic gauge for $t \rightarrow \infty$ where $e_1 = e^{i\theta}e_0$ as usual. Next let $s'_0(t, z)$ be a solution decaying at $t \rightarrow -\infty$ and $s'_1(t, z)$ a complementary solution, i.e. $s'_0(t, z)|t|^{-k/2}e^{|t|} \rightarrow e'_0$.

Combining these solutions into a single scattering process one can write $s'_0(t, z) = a(z)s_0(t, z) + b(z)s_1(t, z)$ where $b(z)$ is a polynomial of degree k which is the degree of the monopole bundle E_k and the charge of the monopole. Next one assigns the rational function $S_m(z) = \frac{a(z)}{b(z)} \bmod b(z)$ to the scattering process driven by the monopole $m = (A, \phi)$. The remarkable theorem of Donaldson holds true:

Theorem 3 ([28]). *The scattering functions $S_m(z), m \in M_k$ are rational functions of degree k .*

There exists a diffeomorphism $M_k \rightarrow R_k$ given by $m \rightarrow S_m(z)$ where R_k is the space of all rational functions with $S_m(\infty) = 0$.

Thus, one can represent $M_k = R_k$ as the space of rational functions in the normal form:

$$S(z) = \frac{\sum_{i=0}^{k-1} a_i z^i}{z^k + \sum_{i=0}^{k-1} b_i z^i}$$

providing the resultant $\Delta(a_0, \dots, a_{k-1}; b_0, \dots, b_k)$ vanishes. This means that M_k is the open set in \mathbb{C}^{2k} . In fact M_k is the *affine algebraic variety*. It is the subvariety in $\mathbb{C}P^{2k}$ given by the complement of the subvariety defined by homogenous equations $b_k \Delta = 0$. The result by Yau then says that M_k has a Ricci-flat Kähler metric.

For $k = 2$ we have

$$S(z) = \frac{a_0 + a_1 z}{z^2 + b_0}, \quad \Delta = a_0^2 + b_0 a_1^2$$

so that the quotient of the algebraic surface in \mathbb{C}^3 : $x^2 - zy^2 = 1$ by the involution $(x, y, z) \rightarrow (-x, -y, z)$ gives the manifold $M_2^0 \subset \mathbb{C}P^1 \times \mathbb{C}^1$ where (a_0, a_1) are homogenous coordinates in $\mathbb{C}P^1$. The great achievement of Atiyah and Hitchin was the explicit description of the metric on M_k^0 . They showed that there exists a Riemannian metric on M_k^0 which is finite, hyperkähler, geodesically complete and $SO(3)$ -symmetric. However, it was the work of Takhtajan [30] showing that this metric is precisely the AH gravitational instanton given in terms of quasi-modular forms.

To show finiteness of the metric one can turn to the zero-modes generating metric as in (24), and investigate their L^2 -norms. This is not an easy task since the equations are defined on the non-compact \mathbb{R}^3 and the gauge freedom should be included. However, Taubes performed the analysis successfully showing the finiteness of the metric of M_k^0 [27].

To find a suitable description of the tangent space modulo gauges, one can turn to the linearization of the Bogomolny equations. For a BPS monopole $m = (A, \phi)$ the tangent space for M_k in m is the space T_m of all pairs (a, ψ) which fulfills the linearized equation [18, 31]:

$$\star D_A a - D_A \psi + [\phi, a] = 0.$$

This should be augmented by the equation

$$\star D_A \star a + [\phi, \psi] = 0$$

which expresses the fact that (a, ψ) is orthogonal to the directions generated by the infinitesimal elements in gauge algebra. Here a is a Lie-algebra-valued 1-form and ψ is a function also with values in the Lie algebra of $SU(2)$. Taubes showed that the dimension of T_m is $4k$. For I, J, K (the base in the quaternion space \mathbf{H}) one can write the 1-form $a = \alpha dx + \alpha dy + \alpha dz$ and ψ , in one expression: $\psi + \alpha I + \beta J + \gamma K$. In this way (a, ψ) becomes a function $(a, \psi) : \mathbb{R}^3 \rightarrow \mathfrak{su}(2) \times \mathbf{H}$. Both equations are invariant with respect to the action of I, J, K , i.e. T_m is the vector space over \mathbf{H} . Now one introduces the natural L^2 -metric on this vector space and I, J, K act as isometries. Given the norm on M_k defined as $\|(a, \psi)\|_m^2 = \|D_A a\|_{L^2}^2 + \|D_A \psi\|_{L^2}^2 + \|[\phi, a]\|_{L^2}^2 + \|[\phi, \psi]\|_{L^2}^2$ one shows that the above metric is complete [27].

$I^2 = J^2 = K^2 = -1$ so there are 3 almost complex structures which are actually integrable (Donaldson) giving rise to 3 complex structures on M_k and to three Kähler forms as well. This means that M_k and M_k^0 are hyperkähler complex manifolds. It is best seen from the infinite-dimensional hyperkähler quotient construction where the 3-components of the Bogomolny equation give rise to the 3-moment maps with values in the dual of the gauge algebra [18]. We remark that the hyperkähler M_k is Ricci-flat.

$SO(3)$ acts isometrically on M_k . This action is given by rotating the complex structures I, J, K on M_k . One projects M_k on \mathbb{R}^3 since the universal cover \tilde{M}_k^0 acts trivially on \mathbb{R}^3 so, the projection is the assignment of the center to a monopole in a $SO(3)$ -equivariant way [18]. We have the metric on M_2^0 which has $SO(3)$ as a group of its isometries. Then we are looking for the solutions of anti-self-dual Einstein equations, which are $SO(3)$ -invariant. The differential equations of this metric in 4-d are reduced to a system of ordinary differential equations. It follows from [4] that the metric written in terms of Maurer-Cartan basis of $SU(2)$ (6), reads:

$$ds^2 = dt^2 + a(\sigma^1)^2 + b(\sigma^2)^2 + c(\sigma^3)^2. \quad (28)$$

Rewriting it in terms of $\Theta^i, i = 1, 2, 3$, such that $dt' = \sqrt{\Theta^1 \Theta^2 \Theta^3} dt$, one arrives at the ordinary differential equations (14) and the case $\Theta^1 \neq \Theta^2 \neq \Theta^3$ is crucial (the $SO(3)$ -orbits are 3-dimensional). That is, the AH geometry of M_2^0 is precisely that of $\mathcal{M}_3 \times \mathbb{R}$ and the metric is written in terms of quasi-modular series E_2 . Thus we close the circle of the argumentation in the paper and arrive at quasi-modularity and the geometry we started with.

Finally let us comment on the BPS condition as allowing for the direct connection of magnetic (charged) monopoles with exotic 4-geometry. Namely, the Lagrangian (18) gives rise to the energy-momentum tensor:

$$T_{\mu\nu} = F_{\mu\rho}^a F_{\nu}^{a\rho} + D_\mu \phi^a D_\nu \phi^a - g_{\mu\nu} L \quad (29)$$

so that the static energy in terms of $E^{ai} = F^{ai0}$ and $B^{ai} = -\frac{1}{2}\epsilon^{ijk} F_{ajk}$ fields reads:

$$E = \int d^3x T_{00} = \int d^3x \frac{1}{2} (E_i^a E_i^a + B_i^a B_i^a + (D_i \phi^a)(D_i \phi^a)) + \frac{\lambda}{4} (\phi^a \phi^a - v^2)^2.$$

For simplicity and due to the absence of explicit formulas on an exotic \mathbb{R}^4 , we assume additionally that this $T_{\mu\nu}$ is precisely generated by the change of the exotic smoothness of e into the standard \mathbb{R}^4 as discussed before. This happens in the FTL limit of GR. Thus, we can consider both, the 'breaking' of the exotic 4-smooth structure which results in the standard one, and the appearing of the Higgs axis with non-zero vev breaking the $SU(2)$ symmetry, as very close related. The appearance of the Higgs direction seen in exotic 4-d, automatically reduces the smoothness into the standard one. Thus, $T_{\mu\nu}$ in the theory with a Higgs is generated by $T_{\mu\nu}$ from an exotic \mathbb{R}^4 . However, the BPS bound (23) gives also charges from masses in the static limit. This purely formal relation indicates the possibility to generate monopoles as charged configurations from exotic 4-geometry. We will discuss this physically intriguing point further in the next section.

The extension of the geometry of M_2^0 over a regime of quantum dynamics of interacting monopoles is possible and was indeed performed [31–33]. This extension shows the true magic of the geometry of M_2^0 which is based on the geodesic approximation to interacting almost static quantum monopoles, the program first proposed by Manton. In particular, the role of the geometry of M_2^0 in the quantum regime is reflected in considering the Schrödinger operator as being proportional to the covariant Laplacian on M_2^0 . Let ξ_1, ξ_2, ξ_3 be the dual vector fields to the Maurer-Cartan 1-forms $\sigma_1, \sigma_2, \sigma_3$ as in (6) for $SU(2)$, and $f = \frac{-\Theta^2}{r}$. Then the Schrödinger equation reads in the coordinates introduced in Sec. IV:

$$-\frac{1}{\Theta^1\Theta^2\Theta^3f}\frac{\partial}{\partial r}\left(\frac{\Theta^1\Theta^2\Theta^3}{f}\frac{\partial\Psi}{\partial r}\right) - \left(\frac{\xi_1^2}{(\Theta^1)^2} + \frac{\xi_2^2}{(\Theta^2)^2} + \frac{\xi_3^2}{(\Theta^3)^2}\right)\Psi = \pi\frac{E}{\hbar^2}\Psi.$$

In certain approximations one obtains the Taub-Nut geometry of quantum interacting monopoles [32]. In fact, this quantum case can be solved exactly. However, it is a merely approximation for the complete quantum dynamics of monopoles in AH metric of M_2^0 , in which case, quantum geodesics show also a chaotic behavior. This last fact can be (possibly) understood in terms of wild embeddings characterizing the exotic 4-geometry. This interesting point will be approached in a separate publication. Moreover, when dealing with quantum aspects of the geometry of moduli spaces there appears a mechanism where angular momentum generates electric charges of the monopoles/dyons configurations [33], which is the case for the energy generating the charges discussed above.

On the other hand, due to the Conjecture V 1 one can approach the regime of quantum gravity via magnetic monopoles. The meaning of this will be discussed elsewhere, however some remarks are placed also in the next section.

VI. DISCUSSION

We obtained a rather remarkable way of introducing the Higgs field into the $SU(2)$ YM theory on 4-d exotic e when changing the smoothness into the standard one. $SU(2)$ magnetic monopoles of topological charge k determine the geometry of their (reduced) moduli spaces M_k^0 . This metric is complete, hyperkähler and defined on the whole M_k^0 and for $k = 2$ matches precisely the geometry of the gravitational instanton emerging from exotic \mathbb{R}^4 . The appearance of this geometry determines the dynamics of low energy magnetic monopoles and is the strong indication for magnetic monopoles as generating the geometry. The metric of it governs the interactions of low energy $k = 2$ magnetic monopoles. Any quantum theory of magnetic $k = 2$ monopoles has to give this geometry in the classical limit [18]. But this geometry governs also the quantum dynamics of monopoles [33]. In our case this geometry is generated by the 4-d exotic e and by the dynamics of GR in the FTL limit. This means that the exotic e captures the limit of the dual magnetic dynamics of point-like monopoles and this can be uniquely prolonged into the regime where monopoles are not point-particles and are rather quantum entities [33]. Given the 4-geometric origins of these monopoles, the Higgs field appears from the classical geometry on e , too. We expect that this can be somehow extended over quantum regimes of the theories: QG and standard model of particles and fields. One indication of this expectation is the relation between magnetic charges of Dirac monopoles and the GV classes of the foliations assigned to the exotic \mathbb{R}^4 s from the radial family of deMichelis-Freedman type [14]. Another hints are derivable from the relation of these exotic \mathbb{R}^4 s with superstring theory. Again foliations and wild embeddings are the crucial geometrical base for the correspondence [10–12]. In this paper we showed how the semiclassical approach to QG is well-suited for analyzing the role of the geometry of e in FTL of GR.

The arguments of this paper indicate the following possibility: starting with the pure $SU(2)$ YM on the exotic \mathbb{R}^4 then changes the smoothness into the standard \mathbb{R}^4 . This can switch on the Higgs field and turn the theory to Yang-Mills-Higgs theory. It breaks exotic smoothness and massive vector bosons appear. The Higgs field breaks the $SU(2)$ symmetry of YM theory on e , but also ϕ breaks the exotic structure of e enforcing the reduction of the $SU(2)$ symmetry to $U(1)$ symmetry along a line in the isospace. The appearance of the Polyakov-'t Hooft magnetic monopoles driven by the 4-geometry can be further discussed from many different physical perspectives, leading to some unifying power.

Cosmology and GUT theories. YMH theory is the modification of the earlier attempt of Georgi and Glashow (1972) to incorporate the Higgs mechanism in the YM theory with $SO(3)$ gauge group. The Polyakov-'t Hooft modification relies on this attempt but with the $SU(2)$ gauge group. This theory does not match the experimental predictions correctly, since the mass of monopoles from YMH would be in the testable range but are not found there. Furthermore the gauge group of SM is not just $SU(2)$. The $SU(5)$ GUT theory (1974) of Georgi and Glashow also predicts the existence of magnetic monopoles of this kind. As is known, the $SU(5)$ GUT predicts instability of proton having a finite live-time which was also falsified experimentally. However, every GUT-type theory with a larger group than $SU(2)$ necessarily predicts the existence of magnetic monopoles when the symmetry is broken and the mass of them can be much higher. Larger groups than $SU(5)$ groups in GUT theories draw this limit and the live-time of proton behind the testable border. The original Dirac monopoles (1931) introduced on the base of quantum mechanics does not have assigned strictly defined masses.

The absence of magnetic monopoles in the universe enforced researchers to find out a reason explaining the low density of monopoles, provided they exist. This led to the theory of inflation, which is the cornerstone of modern cosmology. Here, the possibility appears that monopoles are replaced by 4-geometry which is not limited only to the early universe epoch. So far, there are theoretical arguments that 4-geometry is assigned to condensed effective matter, and this geometry would have physical meaning. Moreover, it is possible that inflation is derived from nonstandard smooth structures on some 4-manifolds. It was described for the case of fake Freedman 'ends' $S^3_\theta \times \mathbb{R}$ [34]. These topics are currently under active development.

Gravity confined to the exotic e , YM and Higgs fields related to magnetic monopoles and to e , all these constituents suggest to consider the theory unifying them, i.e. Einstein-Yang-Mills-Higgs model (EYMH) (see e.g., [35]). This theory introduces essential coupling of Higgs to curvature and is used for approaching some important questions in cosmology and for modeling gravity in condensed matter phenomena (see e.g., [35]). The contributions of the Higgs fields to the total stress-energy tensor of the system are considered in many models as dominating at the inflation stage. The models with Higgs fields appeared in cosmology in the theory of dark matter and dark energy but the discovery of accelerated expansion of the present Universe showed that non-minimal Einstein-Yang-Mills-Higgs models are important in the search for an explanation of the dark energy phenomenon. One derives exact solutions for metrics respecting Higgs background which might be useful in search for metrics on exotic \mathbb{R}^4 s. Moreover, cosmological solutions with naked singularities appear in this framework and naked singularities must also be present in GR on exotic \mathbb{R}^4 [22]. These new attractive approach to exotic 4-spaces in the above context of cosmology will be addressed in a separate publication.

Condensed matter, spin-ice and cold atoms. So far, the search for magnetic monopoles as fundamental particles failed. That is why people started with attempts to realize monopoles as emergent phenomenon, i.e. as manifestations of the correlations present in a strongly interacting many-body system. That is why magnetic monopoles-like states were expected to be found, and are indeed reported as appearing, among states realized on the, so called, spin-ice systems [36–38]. This seems to be a very promising approach which is important also from the exotic 4-geometry point of view. Moreover, one faces growing recent activity on experimental approaching the degenerate quantum gases as revealing the potential of the cold atom systems. They show quite universal features and serve as quantum simulators for ideas far beyond the usual condensed matter phenomena. In particular, non-Abelian gauge potentials could be realized in the effective description of atoms with degenerate internal degrees of freedom coupled to spatially varying laser fields. Namely, dilute Bose-Einstein condensates (BECs) of alkali atoms (with a hyperfine spin degree of freedom) combine magnetic and superfluid order. The order parameter describing such systems is invariant under global symmetries of a non-Abelian group and thus, monopoles can occur if this symmetry is broken. Thus, artificially generated gauge fields in spinor BECs, can provide an alternative method to realize a magnetic monopole and in the simplest case of spin-1 condensate, a variety of different topological defects indeed are expected. These are global monopoles, non-Abelian magnetic monopoles, global textures and analogies to Dirac monopoles [39–41]. An experimental realization of any of these topological states still remains a challenge in the field of cold atoms, though, from the point of view of ideas developed in this paper, the appearance of Dirac and non-abelian 't Hooft-Polyakov magnetic monopoles in the above condensates indicates a new and fundamental connection between 3-d physics and 4-d geometry. The link is given by exotic 4-geometry and the approach deals with the collective states of effective matter. Certainly, the fundamental connection between 4-d and 3-d physics was predicted already by Einstein's relativity theories. However, here the connection is prolonged somehow over the low energy, static regimes, at least for the magnetic monopoles. Previously, based on the algebraic considerations, it was also proposed to relate 4-geometries of certain exotic \mathbb{R}^4 s with the Kondo states created by effective electrons in the multichannel Kondo effect [42]. Notion of fundamental microscopic matter is modified, as is gravity connected with such matter. Even static states of 3-d magnetic matter can be rooted in 4-d non-flat geometries.

The standard model of particles and QG. Magnetic monopoles considered as fundamental point-particles, generates the 4-geometry which has also gravitational meaning seen via the geometry and curvature of exotic \mathbb{R}^4 (cf. [42]). Such situation indicates on the possible two regimes of gravity, one connected with 'microscopic' matter of the standard model and with the standard smoothness on \mathbb{R}^4 , and the other, connected with the effective matter and exotic \mathbb{R}^4 s.

Because of that one could consider two different smoothness structures on which two theories, gravity and SM of particles, are built. This difference is realized in the smoothness assigned to GR and SM. From this point of view one can approach the difficulties of a theory which would unify QFT and GR. Some new ingredients resulting from the approach above might be crucial for unifying the theories (cf. [43]).

Supersymmetry and SW theory. The FTL was recently considered in the context of supersymmetric $\mathcal{N} = 2$ YM theories on e and, in fact, this was the motivation for the present work. Gravitational corrections to Seiberg-Witten theory on \mathbb{R}^4 were determined as the corrections derived from FTL limit of SYM on e [13, 15]. The supersymmetry played a crucial role in the derivation. It is known that SW theory exhibits the confinement and the mass gap but, again, supersymmetry is crucial. One could wonder whether the direct connection of exotic e with magnetic monopoles and the possibility to formulate YM theory on e are sufficient for the generation of confinement mechanism without supersymmetry. The condensation of monopoles would be replaced by 4-geometric structures. If such attempt results in a theory bearing realistic features would be a new insight into the confinement problem. This requires, however, a careful analysis and will be discussed elsewhere. Despite the substantial theoretical effort it would be very desirable and interesting to indicate and analyze purely experimental signals in favor to the role of 4-geometry in various branches of physics, from low energy, particle physics to global gravitational effects. We hope that at near future we will exhibit some of them.

Our knowledge of 3-d geometries became rather complete mainly due to the seminal works by Thurston and Perelman who proved the geometrization conjecture of Thurston. The 4-d case is still mysterious and there remains many unanswered questions especially for open 4-manifolds. The results in this paper as well in some related papers show that certain 4-geometries on \mathbb{R}^4 are able to code dynamics of $3 + 1$ theories with many, also non-perturbative, aspects. These geometries are also assigned to low energy states of condensed matter and may play an active role in redefining fundamental matter and gravity at this regime. This is just the beginning of uncovering the meaning of this code. It seems that much more can be understood in the near future by exploring these questions.

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- [45] One can use also other closed compact 3-manifolds and their foliations.
- [46] In fact, we do not know any of them, but there exist plenty of smooth metrics on every Riemannian smooth manifold.